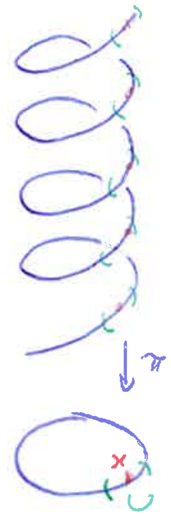
Covering maps:

Def: A covering map between two ~~(topological spaces)~~ Riemann surfaces is a continuous map $p: Y \to X$ so that $\forall x \in X$, $\exists U \ni x$ connected open neighborhood such that any connected component of $p^{-1}(U)$ is mapped by π onto U as an homeomorphism.

A deck transformation associated to the covering π is a continuous map $\gamma : Y \to Y$ satisfying $p \circ \gamma = p$

$$\begin{tikzcd} Y \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Deck transformations are also called automorphisms of the covering, and denoted $Aut(p)$.



Any Riemann surface admits a covering $p : \tilde{X} \to X$ so that $\tilde{X}$ is simply connected ($\pi_1(\tilde{X}) = \{e\}$ $\Leftrightarrow$ Any loop $\gamma : S^1 \to \tilde{X}$ is homotopic to a constant). Such covering is unique up to automorphisms, and called the universal covering of X.

In fact, the universal covering of X exists as far as X is a topological space which is:

- connected ($\nexists\, U, V$ open, $U \cap V = \emptyset$, $U \cup V = X$, $U, V \neq \emptyset$)
- locally path connected ($\forall x \in X$, $\forall U \ni x$ nbhd $\exists\, x \in V \subseteq U$ path connected)
  ($\forall\, y_0, y_1 \in V$ $\exists\, \gamma : [0,1] \to V$ continuous, $\gamma(0) = y_0$, $\gamma(1) = y_1$)
- semi-locally simply connected ($\forall x \in X$ $\exists U \ni x$ open neighborhood s.t. any loop in U is homotopic to a constant within X).

The universal covering of a Riemann surface is itself a Riemann surface, and coverings and deck transformations are holomorphic maps.

A covering p is called regular, or normal, or Galois, if $\mathrm{Aut}(p)$ acts transitively on the fibers of a (every) point:
$\exists x \in X$ (or $\forall x \in X$), $\forall y_1, y_2 \in p^{-1}(x)$, $\exists \gamma \in \mathrm{Aut}(p)$, $\gamma(y_1) = y_2$.

This is equivalent to the condition that $p_* \pi_1(Y, y) \triangleleft \pi_1(X, x)$ (where $p(y) = x$)

normal subgroup

The universal covering is always normal.

In particular, given such universal covering $p: \tilde{X} \to X$, one has that $X \simeq \tilde{X}/\Gamma$, where $\Gamma = \mathrm{Aut}(p)$.

Conversely, given a simply connected Riemann surface $\tilde{X}$ (hence $\tilde{X} = \hat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{D}$), one can ask for which group Γ of automorphisms of $\tilde{X}$, the natural projection $p: \tilde{X} \to \tilde{X}/\Gamma$ gives rise to a normal covering.

It turns out that this happens iff the action $\Gamma \circlearrowright \tilde{X}$ is:

1) free : $\forall \gamma \in \Gamma$, $\gamma \neq \mathrm{id} \Rightarrow \mathrm{Fix}(\gamma) = \emptyset$.

2) properly discontinuous: $\forall K \subset \tilde{X}$ compact, $\#\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\} < \infty$
(K intersects only finitely many of its translates $\gamma(K)$, $\gamma \in \Gamma$.)

Digression on coverings:

3) "regular" action: $\forall x \in \tilde{X}$ $\exists U \ni x$ open neighborhood s.t. $\gamma(U) \cap U = \emptyset$ $\forall \gamma \neq \mathrm{id} \in \Gamma$
(For $\tilde{X}$ Hausdorff)

(1 & 2 $\Rightarrow$ 3) First, notice that condition 2 implies $\forall x \in \tilde{X}$ $\exists U \ni x$ open nbhd s.t. $\gamma(U) \cap U = \emptyset$ for all but finitely many $\gamma \in \Gamma$.

Take such a $U_0 \ni x$, and let $\gamma_1 \ldots \gamma_n$ the elements so that $U_0 \cap \gamma_j(U_0) \neq \emptyset$
$=: U_j$

Set $x_0 = x$; $x_j = \gamma_j(x_0)$ ($\gamma_0 = \mathrm{id}$)

Since the action is free, $\{x_j\}$ are all distinct.

Since $\tilde{X}$ is Hausdorff we may consider $x_j \in V_j \subset U_j$ all distinct open neighborhoods of x_j. Set $W = \bigcap_{j=0}^{n} \gamma_j^{-1}(V_j)$.

Then W is a open neighborhood of x_0 such that $\gamma(W) \cap W = \phi$ $\forall \gamma \in \Gamma$.

In fact $W \subset U_0$, so $\gamma(W) \cap W = \phi$ $\forall \gamma \neq \gamma_0 \dots \gamma_n$.

While $\gamma_j(W) \subset V_j$ which does not intersect $V_0 \overset{\supset W}{}$ by construction.

($3 \Rightarrow 1$) easy (trivial)

($3 \Rightarrow 2$) K compact. $\forall x \in K$ let U_x be such that $\gamma(U_x) \cap U_x \neq \phi \Rightarrow \gamma = \mathrm{id}$.

$\{U_x\}_{x \in K}$ is a open covering of $K \rightsquigarrow$ extract a finite covering $\{U_{x_1} \dots U_{x_n}\}$

Claim: $\forall j$ $\exists$ finitely many γ so that $\gamma(U_{x_j}) \cap K \neq \phi$.

For $U = U_{x_j}$ consider $D = \{x_j\} \subset U$ closed, $W = \tilde{X} \setminus \bigcup_\gamma \gamma(D)$.

We claim that W is open. If this is true then $\{\gamma(U) \mid \gamma \in \Gamma\} \cup \{W\}$ is a open covering of K. $\rightsquigarrow$ We may extract a finite covering of K $\{W, \gamma_1(U) \dots \gamma_2(U)\}$

Since $\forall \gamma$ $\exists p_\gamma \in \gamma(U) \setminus \bigcup_{\delta \neq \gamma} \delta(U)$ we deduce that $\gamma(U) \cap K = \phi$ $\forall \gamma \neq \gamma_1 \dots \gamma_2$.

To prove that W is open: For any $p \in W$: take U_p so that $U_p \ni p$. $\gamma(U_p) \cap U_p = \phi$ $\forall \gamma \neq \mathrm{id}$.

Then $\exists$ at most one γ so that $U_p \ni \gamma(x_j)$. Set $V_p = U_p \setminus \{\gamma(x_j)\}$, which is still open, $p \in V_p$ and $V_p \subset W$. $\Rightarrow W$ is open. The claim is proved.

To sum up: $\forall j$ $\exists$ finitely many γ so that $\gamma(U_j) \cap K \neq \phi$.

Taking all such γ for $j = 1 \dots N$ we get finitely many γ so that $\gamma(K) \cap K \neq \phi$ and we are done. $\square$

Γ acts freely and properly discontinuously on $\tilde{X} \Leftrightarrow pr: \tilde{X} \to X$ is a regular covering. (2.16)

Proof: ($\Rightarrow$) $pr: \tilde{X} \to X = \tilde{X}/_\Gamma$ is surjective by construction.

Given U open in $\tilde{X}$, $pr(U)$ is open $\Leftrightarrow$ $pr^{-1}(pr(U))$ is open in $\tilde{X}$.

But $pr^{-1}(pr(U)) = \bigcup_{\gamma \in \Gamma} \gamma(U)$, which is open because γ are homeomorphisms

Let $x \in \tilde{X}$ be any point, and $U_x \ni x$ a neighborhood so that $\gamma(U_x) \cap U_x = \phi$

$\forall \gamma \in \Gamma \setminus \{id\}$ $\Rightarrow$ $\forall \gamma \neq \delta$ $\gamma(U_x) \cap \delta(U_x) = \phi$.

(if $\phi \neq \delta(U_x) \cap \gamma(U_x) \Rightarrow \phi \neq \gamma^{-1}(\gamma(U_x) \cap \delta(U_x)) = U_x \cap \gamma^{-1}\delta(U_x)$)

$\Rightarrow pr^{-1}(pr(U_x))$ is the disjoint union of $\gamma(U_x)$ $\forall \gamma \in \Gamma$

Notice that $\forall u \in U_x$, $U_x \cap \underset{\{\gamma(u) \mid \gamma \in \Gamma\}}{[u]_\Gamma} = \{u\}$ or it would contradict $U_x \cap \gamma(U_x) \neq \phi$ $\forall \gamma \in \Gamma \setminus \{id\}$

[Similarly for its translates $\gamma(U_x)$]

Hence $\forall u, v \in U_x$, $pr(u) = pr(v) \Leftrightarrow [u]_\Gamma = [v]_\Gamma \Rightarrow u = v$, and

$pr|_{U_x}: U_x \to pr(U_x)$ is injective (hence a bijective continuous map)

Since pr is also open, $pr|_{U_x}$ is a homeomorphism.

It follows that pr is a covering map

Since $\forall x, y$ s.t. $[x]_\Gamma = [y]_\Gamma$ there exists $\gamma \in \Gamma$, $\gamma(x) = y$ we deduce

that pr is a regular covering (and $Aut(pr) \underset{(=)}{\supseteq} \Gamma$)

($\Leftarrow$) We show that if pr is a normal covering then the group of deck transformations

$Aut(pr) = \Gamma$ acts freely and properly discontinuously on $\tilde{X}$.

In fact, this happens for any covering, while the normality ensures

that $X = \tilde{X}/_\Gamma$.

To see this $\forall x \in \tilde{X}$ let $U \ni pr(x)$ be a evenly covered (connected) nbhd of $p = pr(x)$.

Then $pr^{-1}(p) = \bigcup_{pr(y)=p} U_y$, where U_y are connected open containing y,

and $pr: U_y \to U$ is a homeomorphism

(2.11)

Then U_x satisfies $\gamma(U_x) \cap U_x \neq \emptyset \Rightarrow \gamma = \mathrm{id}$

In fact if $\gamma(U_x) \cap U_x \neq \emptyset \Rightarrow \gamma(U_x) \subseteq U_x$ (by connectivity) and by $pr \circ \gamma = pr$ we get $\gamma = \mathrm{id}$ $\left(\gamma = (pr|_{U_x})^{-1} \circ pr = \mathrm{id}\right)$.

The map $pr : \tilde{X} \to X$ descends to a continuous map $\Phi : \tilde{X}/_\Gamma \to X$, since $pr = pr \circ \gamma \quad \forall \gamma \in \Gamma$.

Moreover, $\Phi([x]_\Gamma) = \Phi([y]_\Gamma) \Leftrightarrow$ ~~..~~ $pr(x) = pr(y)$.

By normality $\exists \gamma \in \Gamma$ s.t. $\gamma(x) = y$ and $[x]_\Gamma = [y]_\Gamma$. hence Φ is injective (and hence a continuous bijection).

Being pr open, Φ is open too hence an homeomorphism and $X \cong \tilde{X}/_\Gamma$ □

Hence, to describe non simply connected Riemann surfaces we need to study (discrete) subgroups of $\mathrm{Aut}(\tilde{X})$ acting freely and properly discontinuously, where $\tilde{X} = \hat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$ (Aut = biholomorphisms).

a few properties of holomorphic functions

Lemma (Schwarz) Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map so that $|f(z)| \leq 1 \quad \forall z \in \mathbb{D}$ and $f(0) = 0$.

Then $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$ and $|f'(0)| \leq 1$.

Moreover, if equality holds for some $z_0 \in \mathbb{D}^*$, or if $|f'(0)| = 1$ then $f(z) = cz$ for some constant c, $|c| = 1$.

Proof: Consider the map $g(z) = \frac{f(z)}{z}$. Since $f(0) = 0$, g has a removable singularity at 0, and extends to a holomorphic map $g : \mathbb{D} \to \mathbb{C}$ by setting $g(0) = f'(0)$.

Let $D(0,r)$ be the open disc centred at 0 of radius $r < 1$.

By the maximum principle, $\underset{\forall z \in D(0,r)}{|g(z)|} \leq \max_{z \in \partial D(0,r)} |g(z)| = \max_{z \in \partial D(0,r)} \frac{|f(z)|}{r} \leq \frac{1}{r}$.

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By letting $r \to 1^-$, we infer $|f(z)| \leq z$ and $|f'(0)| \leq 1$ $\forall z \in \mathbb{D}$.

Suppose there exists $z_0 \in \mathbb{D}$ so that $|g(z_0)| = 1$.
Then $|g|$ admits a local max ~~point~~ at z_0, and $g \equiv c$ is constant. (with $|c| = 1$).
Hence $f(z) = cz$. $\square$

Proposition ($\mathrm{Aut}(\mathbb{D})$). $\mathrm{Aut}(\mathbb{D}) = \left\{ z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z} \;\middle|\; \theta \in \mathbb{R},\ a \in \mathbb{D} \right\}$.

Proof. Set $f_a : \mathbb{D} \to \mathbb{C}$, $z \mapsto \frac{z-a}{1-\bar{a}z}$.

First notice that $f_a \in \mathrm{Aut}(\mathbb{D})$. In fact $1 - \bar{a}z = 0 \iff z = \frac{1}{\bar{a}} \notin \mathbb{D}$, and

$\frac{z-a}{1-\bar{a}z} \in \mathbb{D} \iff \left| \frac{z-a}{1-\bar{a}z} \right| < 1 \iff |z-a| < |1-\bar{a}z| \iff (z-a)(\overline{z-a}) < (1-\bar{a}z)(1-a\bar{z})$

$\iff z\bar{z} - \cancel{a\bar{z}} - \cancel{\bar{a}z} + a\bar{a} < 1 - \cancel{a\bar{z}} - \cancel{\bar{a}z} + a\bar{a}z\bar{z} \iff |z|^2 \cdot |a|^2 - |z|^2 - |a|^2 + 1 > 0$

$\iff (1-|z|^2)(1-|a|^2) > 0$ Since $|a| < 1$, $\iff |z| < 1$. ok.

Being f_a a rational map of degree 1, it is injective and hence $f_a \in \mathrm{Aut}(\mathbb{D})$. One can also check that $f_a \circ f_{-a} = f_{-a} \circ f_a = \mathrm{Id}$.

Notice that $f_a(a) = 0$.

Let now f be any element of $\mathrm{Aut}(\mathbb{D})$, and let $a = f^{-1}(0)$.

Then $g = f \circ f_a^{-1} \in \mathrm{Aut}(\mathbb{D})$ and $g(0) = 0$.

By Schwarz lemma, $|g(z)| \leq |z|$ $\forall z \in \mathbb{D}$.

By Schwarz lemma applied to g^{-1}, we get $|g^{-1}(z)| \leq z$ $\forall z \in \mathbb{D}$, and $|g(z)| = |z|$ $\forall z \in \mathbb{D}$. Again by Schwarz lemma, $g(z) = e^{i\theta} z$, $\theta \in \mathbb{R}$. $\square$

Rem $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$:

$\forall a, b \in \mathbb{D}$ $\exists f \in \mathrm{Aut}(\mathbb{D})$, $f(a) = b$. Here $f = f_b^{-1} \circ f_a$.

One could say a little more: could ~~suppose~~ bring a to 0 and b into $(0,1)$ (for example)

Poincaré half-plane ↖ It acts doubly transitively on $\partial\mathbb{D}$.

Set $\mathbb{H} = \{x + iy \mid y > 0\} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$

Then $\mathbb{H}$ is biholomorphic to $\mathbb{D}$:

$\Phi : \mathbb{D} \to \mathbb{H}$, $\quad \Phi(z) = i \cdot \dfrac{1-z}{1+z}$. $\qquad \Phi^{-1}(w) = \dfrac{i - w}{i + w}$.

It is easy to check that $\Phi \circ \Phi^{-1} = \mathrm{id}$.

Moreover: $\mathrm{Im}\, \Phi(z) = \mathrm{Re}\left(\dfrac{1-z}{1+z}\right) = \dfrac{1}{2}\left(\dfrac{1-z}{1+z} + \dfrac{1-\bar z}{1+\bar z}\right) = \dfrac{1 - |z|^2}{|1+z|^2} > 0 \iff |z| < 1$.

In this model,

$\mathrm{Aut}(\mathbb{H}) = \left\{ z \mapsto \dfrac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad - bc > 0 \right\}$

$\simeq \mathbb{P}GL(2, \mathbb{R})$

In fact, Notice that $\mathrm{Aut}(\mathbb{D})$ is generated by $\{z \mapsto \lambda z \mid |\lambda| = 1\} \cup F$,

where F is any family in $\mathrm{Aut}(\mathbb{D})$ so that $\forall a \in \mathbb{D}$ $\exists f \in F$, $f(a) = 0$

(or $f(0) = a$)
(or $f(a) = b$)

Notice that $G = \left\{ z \mapsto \dfrac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad - bc > 0 \right\}$ is a group,

a subgroup of $\mathrm{Aut}(\mathbb{H})$, since:

$\mathrm{Im}\, \dfrac{az+b}{cz+d} = \dfrac{1}{2i}\left(\dfrac{az+b}{cz+d} - \dfrac{a\bar z + b}{c\bar z + d}\right) = \dfrac{1}{2i}\left(\dfrac{(az+b)(c\bar z + d) - (a\bar z + b)(cz+d)}{|cz+d|^2}\right) \iff$

$(\forall z \in \mathbb{H})$

$0 < \dfrac{1}{2i}(ad - bc)(z - \bar z) = (ad - bc)\, \underset{\substack{\vee \\ 0}}{\mathrm{Im}\, z}$ since $z \in \mathbb{H}$. $\Rightarrow \dfrac{az+b}{cz+d} \in \mathbb{H}$, and $G < \mathrm{Aut}(\mathbb{H})$.

G acts transitively on $\mathbb{H}$, since $z \overset{f}{\mapsto} az + b$ belongs to G $\forall a, b \in \mathbb{R}$, $a > 0$,

and $f(i) = ai + b$ is any point in $\mathbb{H}$.

The rotations $\rho_\theta : z \mapsto \underbrace{e^{i\theta}}_{\lambda} z$ correspond to $\Phi \circ \rho_\theta \circ \Phi^{-1} : w \mapsto \dfrac{i(\lambda+1) z + \lambda - 1}{i(\lambda+1) z + 1 - \lambda}$

Connected to the matrix $\begin{pmatrix} i(\lambda+1) & \lambda-1 \\ 1-\lambda & i(\lambda+1)\end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i\end{pmatrix}$)

This is equivalent to $w \mapsto \dfrac{w\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{-w\sin\frac{\theta}{2}+\cos\frac{\theta}{2}} \in G$.

In fact, set $\mu = e^{i\frac{\theta}{2}}$, then $\begin{pmatrix} i(\mu^2+1) & \mu^2-1\\ 1-\mu^2 & i(\mu^2+1)\end{pmatrix} \sim \begin{pmatrix}\frac{\mu+\mu^{-1}}{2} & \frac{\mu-\mu^{-1}}{2i}\\ \frac{\mu^{-1}-\mu}{2i} & \frac{\mu+\mu^{-1}}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac\theta2 & \sin\frac\theta2\\ -\sin\frac\theta2 & \cos\frac\theta2\end{pmatrix}$ □

Fixed points of elements of $\mathrm{Aut}(\mathbb{D})$

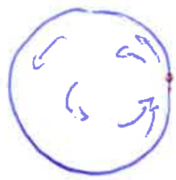
$f(z) = \lambda\cdot\dfrac{z-a}{1-\bar a z}$ $\quad a\in\mathbb{D}$, $|\lambda|=1$. assume $a\neq 0$.

$z = f(z) \Leftrightarrow z(1-\bar a z) = \lambda(z-a) \Leftrightarrow \bar a z^2 + z(\lambda-1) - a\lambda = 0$.

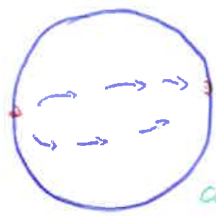
In particular, there are always two fixed points when f is seen on $\hat{\mathbb{C}}$, as if $a=0$, these points are 0 and ∞. If $a\neq 0$, then $z_1\cdot z_2 = \dfrac{-a\lambda}{\bar a}$ so $|z_1 z_2| = \dfrac{|a\lambda|}{|\bar a|} = 1$.

There are three possible cases:

- $\exists!\, z\in\mathbb{D}$, $z=f(z)$ (elliptic case).



- $\exists!\, z\in\partial\mathbb{D}$ $z=f(z)$ (parabolic case)



- $\exists\, z_1\neq z_2\in\partial\mathbb{D}$ satisfying $z=f(z)$ (hyperbolic case)

Examples: elliptic case: $z\mapsto e^{i\theta}z$ (rotation with center 0) corresponds to $\dfrac{\cos\frac\theta2 w+\sin\frac\theta2}{-\sin\frac\theta2 w+\cos\frac\theta2}$ on $\mathbb{H}$

hyperbolic case: $z \overset{f_a}{\mapsto} \dfrac{z-a}{1-az}$, $a\in(-1,1)\setminus\{0\}$ $\quad$ (Fix $=\{-1,1\}$, if $a=\tanh(t)$, then $f_a^2 = f_{\tanh(2t)}$.)

corresponds to the family $F(w)=\mu w$, $\mu\in\mathbb{R}_+^*\setminus\{1\} = (0,+\infty)\setminus\{1\}$

parabolic case $z\mapsto (1+2\bar a)\dfrac{z-a}{1-\bar a z}$, with a satisfying $|a+\frac12| = \frac12$ $(a\neq 0,-1)$

corresponds to the family $F(w) = w+b$, $b\in\mathbb{R}\setminus\{0\}$

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Rem: Notice that if $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ acts freely (and properly ~~discontin~~ discontinuously), it cannot contain any elliptic element.

We now focus on $\mathbb{C}$ and $\hat{\mathbb{C}}$. Rem: We call any surface covered by $\mathbb{D}$ "hyperbolic".

Lemma (Cauchy Derivative Estimate). If f holomorphic maps a disc $\overline{D(z_0,r)}$ into some disc $\overline{D(w_0,s)}$, then $|f'(z_0)| \leq s/r$.

Proof: set $g(z) = f(z_0+z) - w_0$, so that $g(\overline{D(0,r)}) \subseteq \overline{D(0,s)}$.

By the Cauchy integral formula, $\forall r' < r$, we get

$$|f'(z_0)| = |g'(0)| = \left| \frac{1}{2\pi i} \oint_{|z|=r'} \frac{g(z)}{z^2} dz \right| \leq \frac{1}{2\pi} \cdot \frac{\sup |g(z)|}{(r')^2} \cdot 2\pi r' = \frac{s}{r'} \underset{r' \to r}{\longrightarrow} \frac{s}{r}$$ □

Alternative proof: $f: \mathbb{D} \to \mathbb{D}$, $f(0) = a$, $g(z) = f_a \circ f$ sends 0 to 0.

$$g'(0) = f_a'(a) \cdot f'(0) \Rightarrow |f'(0)| = \underbrace{|g'(0)|}_{\substack{\leq 1 \\ \text{Schwarz}}} \cdot \underbrace{|f_a'(a)|^{-1}}_{= 1-|a|^2 \leq 1} \leq 1.$$

Given now f such that $f(\overline{D(z_0,r)}) \subseteq \overline{D(w_0,s)}$ and

$$g(z) = \frac{1}{s}\left(f(z_0 + rz) - w_0\right)$$ so that $g(\mathbb{D}) \subseteq \mathbb{D}$, we get

$$|g'(0)| \leq 1 \quad \text{and} \quad g'(0) = \frac{r}{s} f'(z_0) \Rightarrow |f'(z_0)| \leq \frac{s}{r}.$$ □

Consequence: Liouville theorem: $f: \mathbb{C} \to \mathbb{C}$ entire and bounded $\Rightarrow$ f is constant.

Proof. $\exists s >> 0$ s.t. $f(\mathbb{C}) \subset \overline{D(0,s)}$. Apply lemma $\forall z_0$ $\forall r > 0$ $(r \to \infty)$

$\Rightarrow |f'(z_0)| \leq 0$ $\forall z_0$ $\Rightarrow$ $f' \equiv 0$ $\Rightarrow$ $f = $ const. □

Corollary: $f: \hat{\mathbb{C}} \to \mathbb{C}$ holomorphic $\Rightarrow$ f is constant

Given $f: \mathbb{C} \to \mathbb{C}$ entire, we can see it as a map $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, with a singularity at ∞.

We have three cases:

- ∞ is a removable singularity: $\exists \lambda = \lim_{z\to\infty} f(z) \in \mathbb{C}$.

  $\Rightarrow$ f is bounded, and hence constant.

- ∞ is a pole. Prop: in this case, f is a polynomial (of degree the order of pole at ∞).

Proof: Since $\lim_{z\to\infty} |f(z)| = +\infty$, $f(z) \neq 0$ outside a compact. Being zeroes isolated, we get that f has finitely many zeroes $z_1 \dots z_s$, with multiplicities $m_1 \dots m_s$. Set $P(z) = (z-z_1)^{m_1} \cdot \ldots \cdot (z-z_s)^{m_s}$.

Set $g(z) = \dfrac{f(z)}{P(z)}$. g is holomorphic, defined on all $\mathbb{C}$ (entire), and $\neq 0$.

At ∞, g has either a removable or pole singularity. Hence either g or $\frac{1}{g}$ is a holomorphic function from $\hat{\mathbb{C}} \to \mathbb{C}$, hence it is constant. $\square$

- ∞ is an essential singularity. In this case, we say that f is transcendental. Examples e^z, $\cos z$

Prop: $\mathrm{Aut}(\mathbb{C}) \left(= \mathrm{Aff}^*(\mathbb{C})\right) = \{ z \mapsto az+b \mid a \in \mathbb{C}^*, b \in \mathbb{C} \}$

Proof: $f \in \mathrm{Aut}(\mathbb{C})$ cannot have an essential singularity at ∞ (or f wouldn't be injective), hence $f \in \mathbb{C}[z]$, and it is invertible $\Leftrightarrow \deg f = 1$. $\square$

Rem: $\underset{\mathrm{Aut}(\mathbb{C})}{\mathrm{Fix}\,(f)} = \emptyset \Leftrightarrow f$ is a translation: $f(z) = z + b$ $(b \in \mathbb{C}^*)$

In particular, any $\Gamma < \mathrm{Aut}(\mathbb{C})$ acting freely must be generated by translations. If it acts also properly discontinuously, it must be discrete, hence generated by either $\{0\}$, 1 or 2 translations.
We get $X = \mathbb{C}/\Gamma$ to be $\mathbb{C}$, $\mathbb{C}^*$ ($\sim$ cylinder) or a torus. $\mathbb{T} = \mathbb{C}/\langle 1,\tau\rangle$
(These are the "parabolic" Riemann surfaces)

We conclude with the Riemann sphere.

Prop: Let X be a Riemann surface. Then $f : X \to \hat{\mathbb{C}}$ is holomorphic $\iff$ $f : X \setminus f^{-1}(\infty) \to \mathbb{C}$ is meromorphic.

Proof: Being the properties local, we may assume $X = D(0,r)$ and $f^{-1}(\infty) = \{0\}$. $f : X \to \hat{\mathbb{C}}$ holomorphic at 0 means that $\frac{1}{f} : X \to \mathbb{C}$ (may also assume $f(z) \neq 0$ $\forall z \in D(0,r)$) is holomorphic.

In particular this happens $\iff$ $\frac{1}{f(z)} = z^m \cdot g(z)$ g holomorphic and $g(0) \in \mathbb{C}^*$.
But then $\overset{\iff}{}$ $f(z) = z^{-m} \cdot \frac{1}{g(z)}$ has a pole at 0. □

Prop: $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ holomorphic $\iff$ f is rational ($f \in \mathbb{C}(z)$)

Proof: Being $\hat{\mathbb{C}}$ compact $f^{-1}(0) \cup f^{-1}(\infty)$ is a finite set (f has isolated zeroes and poles = zeroes of $\frac{1}{f}$) $\mathbb{C} \cap f^{-1}(0) = \{z_1 \dots z_r\}$ with multiplicities $m_1 \dots m_r$, $\mathbb{C} \cap f^{-1}(\infty) = \{w_1 \dots w_s\}$ with multiplicities $n_1 \dots n_s$.

Set $R(z) = \dfrac{\prod_{j=1}^{r} (z-z_j)^{m_j}}{\prod_{j=1}^{s} (z-w_j)^{n_j}}$. Then $\dfrac{f(z)}{R(z)} =: g(z)$, $g : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{0,\infty\}$ is holomorphic. It extends to $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ holomorphic and avoiding either 0 or ∞. Hence g or $\frac{1}{g}$ are holomorphic $\hat{\mathbb{C}} \to \mathbb{C}$, hence constant □

Prop. $\mathrm{Aut}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \right\}$

↑ Möbius transformation

Proof. $f \in \mathrm{Aut}(\hat{\mathbb{C}}) \Rightarrow f = \dfrac{P(z)}{Q(z)}$ (P, Q without common factors)

Let $\deg(f) := \max\{\deg P, \deg Q\}$.

Then for any $w \in \hat{\mathbb{C}}$, $f^{-1}(w)$ has $\deg f$ elements counted with multiplicity: they satisfy $P(z) - wQ(z) = 0$.

(ok for generic w. If w is s.t. there is a drop in the degree of $P - wQ$, it means that $f^{-1}(w) \ni \infty$, with multiplicity equal to the drop of degree).

$\Rightarrow \deg f = 1$, i.e., f is a Moebius transformation. $\square$

Rem: $\forall f \in \mathrm{Aut}(\hat{\mathbb{C}})$, $\mathrm{Fix}(f) \neq \emptyset$ ($\#\mathrm{Fix}(f) = 1$).

$\Rightarrow$ The only group of automorphisms acting freely on $\hat{\mathbb{C}}$ is $\{id\}$, and the only Riemann surface covered by $\hat{\mathbb{C}}$ is $\hat{\mathbb{C}}$ itself.] only example of "elliptic" surface

Prop: Let X, Y be two Riemann surfaces, and $f: X \to Y$ holomorphic.

If: X is $\hat{\mathbb{C}}$ and Y is parabolic / hyperbolic;
X is parabolic and Y is hyperbolic] then f is constant.

Proof: Consider the universal coverings of X and Y:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p_X"'] \arrow[dr, "g"] & \tilde{Y} \arrow[d, "p_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

f lifts to $\tilde{f}$ holomorphic (since $\tilde{X}$ is simply connected)

($\Rightarrow$ we may assume X, Y simply connected)

In all situations, we get $\tilde{f}|_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ bounded (by construction if $\tilde{Y} = \mathbb{D}$, and by compactness if $\tilde{X} = \hat{\mathbb{C}}$).

By Liouville, $\tilde{f}|_{\mathbb{C}}$ is constant. It follows that f is itself constant. (covering maps are local homeomorphisms) □

Theorem (Little Picard Theorem). Any connected non-empty open subset U of $\hat{\mathbb{C}}$ avoiding at least three points of $\hat{\mathbb{C}}$ (i.e. contained in $\hat{\mathbb{C}} \setminus \{a,b,c\}$) is a hyperbolic Riemann surface.

Proof. U cannot be elliptic, since the only open compact subsets of $\hat{\mathbb{C}}$ are $\emptyset$ and $\hat{\mathbb{C}}$ itself.

We first show that $\hat{\mathbb{C}} \setminus \{a,b,c\}$ is hyperbolic.

In fact, the fundamental group of $\hat{\mathbb{C}} \setminus \{a,b,c\}$ is not abelian, while all parabolic and elliptic Riemann surfaces have abelian fundamental group.

If $p : \mathbb{C} \to U$ is a covering, then $f \circ p : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{a,b,c\}$ is a holomorphic map between a parabolic on a hyperbolic surfaces. Hence it is constant, which is a contradiction. □

Other properties:

$\mathrm{Aut}(\mathbb{C})$ is simply 2-transitive

$\mathrm{Aut}(\hat{\mathbb{C}})$ is simply 3-transitive.

Def. Γ acts (simply) n-transitive on X if $\forall (x_1 \ldots x_n), (y_1 \ldots y_n) \in X^n$ ~~distinct~~ n-ple of distinct points, $\exists (!) \gamma \in \Gamma$ s.t. $\gamma(x_j) = y_j$ $\forall j = 1 \ldots n$.

Proof: The property for $\mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}^{+}(\mathbb{C})$ is easy.

2.20

We want to show that $\forall\, \alpha, \beta, \gamma$ distincts, $\exists\, f(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ such that $f(0) = \frac{b}{d} = \alpha$; $f(1) = \frac{a+b}{c+d} = \beta$, $f(\infty) = \frac{a}{c} = \gamma$.

If $\gamma = \infty$, $f \in Aut(\mathbb{C})$, and the property follows from the 2-transitivity of $Aut(\mathbb{C})$

If $\gamma \in \mathbb{C}$ $\Rightarrow$ $c \neq 0$, $a = \gamma c$.

If $\alpha = \infty$ $\Rightarrow$ $d = 0$, $b \neq 0$, and $\frac{\gamma c + b}{c} = \beta$ has a unique solution (up to ~~[illegible]~~ multiplication by scalars on (a,b,c,d)) since $\beta \neq \gamma$.

If γ and $\alpha \neq \infty$. a direct computation gives $f(z) = \frac{\gamma z - \alpha}{z-1}$ if $\beta = \infty$.

If $\alpha, \beta, \gamma \neq \infty$, the system $\begin{cases} b - \alpha d = 0 \\ a + b - \beta c - \beta d = 0 \\ a - \gamma c = 0 \end{cases}$ has rank 1, and we conclude. $\square$

Rem: Holomorphic maps can be seen as ramified (or branched) coverings.

If $f: X \to Y$ is a holomorphic map between Riemann surfaces, locally at some point $p \in X$ and at $q = f(p) \in Y$, one can write

~~[illegible]~~ $\Rightarrow$ $f(z) = z^m \cdot g(z)$, $g(z) \neq 0$. (z local parameter at p, w local parameter at $f(p) = q$)

Since $g(z) \neq 0$, $\exists\, h$ holomorphic in a neighborhood of p s.t. $h(z)^m = g(z)$, and $f(z) = (\underbrace{z\, h(z)}_{\phi(z)\ \text{local parameter}})^m$. In particular $f \circ \phi^{-1}(z) = z^m$

$\Rightarrow$ Up to changing the local parameter at p, f can be written as a power

If $m = 1$, the map defines a local diffeomorphism. If $m > 1$, p is called a ~~branched~~ point (or critical point) of order m, and $f(p)$ a ramification point (critical values)

If f is proper ($\Leftarrow X$ compact), $f|_{X \setminus f^{-1}f(e_p)} : X \setminus f^{-1}(f(e_p)) \to f(X) \setminus f(e_p)$ is a covering map Y